

Relaxation in time elapsed neuron network models in the weak connectivity regime

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Abstract

In order to describe the firing activity of a homogenous assembly of neurons, we consider time elapsed models, which give mathematical descriptions of the probability density of neurons structured by the distribution of times elapsed since the last discharge. Under general assumption on the firing rate and the delay distribution, we prove the uniqueness of the steady state and its nonlinear exponential stability in the weak connectivity regime. In other words, total asynchronous firing of neurons appears asymptotically in large time. The result generalizes some similar results obtained in [16, 17] in the case without delay. Our approach uses the spectral analysis theory for semigroups in Banach spaces developed recently by the first author and collaborators.

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1 Introduction

In nervous systems, neuronal circuits carry out tasks of information transmission and processing. Many neurons generate trains of stereotyped electrical pulses in response to incoming stimulations. Following each discharge, the neuron undergoes a period of refractoriness during which it is less responsive to inputs, before recovering its excitability [17]. The main carrier of information is the discharge times or some statistics of the discharge times. In this work, we consider a simple neuronal model which neglects the mechanisms underlying spike generation and focusses on describing the neuronal dynamics in terms of discharge times. More precisely, we consider a model which has been introduced and studied in [7, 16, 17] and which describes the post-discharge recovery of neuronal membranes through an instantaneous firing rate that depends on the time elapsed since the last discharge and the inputs by neurons. We refer to these papers for biologic motivation and discussions. We also refer to [2, 5, 22, 20] where these models (or similar ones) are obtained as a mean field limit of finite number of neuron network models.

The neuronal network is described here by the density number of neurons $f = f(t, x) \geq 0$ which at time $t \geq 0$ are in the state $x \geq 0$. The state of a neuron is a local time (or internal clock) which corresponds to the elapsed time since the last discharge. The dynamic of the neuron network is given by the following nonlinear time elapsed (or of age structured type) evolution

equation

$$\partial_t f = -\partial_x f - a(x, \varepsilon m(t))f =: \mathcal{L}_{\varepsilon m(t)} f, \quad (1.1a)$$

$$f(t, 0) = p(t), \quad f(0, x) = f_0(x). \quad (1.1b)$$

Here $a(x, \varepsilon \mu) \geq 0$ represents the firing rate of a neuron in the state x for a network activity $\mu \geq 0$ and a network connectivity parameter $\varepsilon \geq 0$. The function $p(t)$ represents the total density of neurons which undergo a discharge at time t and is defined through

$$p(t) := \mathcal{P}[f(t); m(t)], \quad (1.2)$$

where

$$\mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx. \quad (1.3)$$

The function $m(t)$ represents the network activity at time $t \geq 0$ resulting from earlier discharges and is defined by

$$m(t) := \int_0^\infty p(t-y)b(dy), \quad (1.4)$$

where the delay distribution b is a probability measure which takes into account the persistence of the electric activity in the network resulting from discharges (synaptic integration). In the sequel, we will consider the two following situations :

- The *case without delay*, when $b = \delta_0$ and then $m(t) = p(t)$.
- The *case with delay*, when b is a smooth function.

We observe that in both cases, the solution f of the time elapsed equation (1.1)–(1.3) satisfies

$$\frac{d}{dt} \int_0^\infty f(t, x) dx = f(t, 0) - \int_0^\infty a(x, \varepsilon m(t)) f(t, x) dx = 0.$$

As a consequence, the total density number of neurons (also called *mass* in the sequel) is conserved and we can normalize that mass to be 1. In other words, we may always assume

$$\langle f(t, \cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) dx.$$

A (normalized) steady state for the time elapsed evolution system of equations (1.1)–(1.3) is a couple $(F_\varepsilon, M_\varepsilon)$ of a density number of neurons $F_\varepsilon = F_\varepsilon(x) \geq 0$ and a network activity $M_\varepsilon \geq 0$ such that

$$0 = -\partial_x F_\varepsilon - a(x, \varepsilon M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \quad (1.5a)$$

$$F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \quad (1.5b)$$

It is worth emphasizing that for a steady state the associated network activity and discharge activity are two equal constants because of the normalization of the delay distribution, i.e. $\langle b \rangle = 1$.

In equations (1.1)–(1.3) and (1.5), the connectivity parameter $\varepsilon \geq 0$ corresponds to the strength of the influence of the neuronal network activity on each neuron through the functions $m(t)$ and $f(t, x)$ respectively. In the limit case $\varepsilon = 0$, equation (1.1)–(1.3) is linear which means that each neuron evolves accordingly to its own dynamic. In the other hand, when $\varepsilon > 0$, equation (1.1)–(1.3) is nonlinear and the dynamic of any given neuron is affected by the state (or the past states in the case of the model with delay) of all the other neurons through the global activity of the neuronal network. Finally, the weak connectivity regime, about which we are mainly concerned in the present paper, corresponds to a range of connectivity parameter $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$ small enough, such that the nonlinearity of equations (1.1)–(1.3) and (1.5) is not too strong.

Our main purpose in this paper is to prove that solutions to the time elapsed evolution equation (1.1)–(1.3) converge to a stationary state under a weak connectivity assumption. Before stating that result, let us present the precise mathematical assumptions we will need on the firing rate a and on the delay distribution b .

We make the physically reasonable assumption

$$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0, \quad (1.6)$$

$$0 < a_0 := \lim_{x \rightarrow \infty} a(x, 0) \leq \lim_{x, \mu \rightarrow \infty} a(x, \mu) =: a_1 < \infty, \quad (1.7)$$

as well as the smoothness assumption

$$a \in W^{2,\infty}(\mathbb{R}_+^2). \quad (1.8)$$

In the delay case, we assume that $b(dy) = b(y) dy$ satisfies the exponential bound and smoothness condition

$$\exists \delta > 0, \quad \int_0^\infty e^{\delta y} (b(y) + |b'(y)|) dy < \infty. \quad (1.9)$$

We begin by stating our main result about the stationary problem (1.5).

Theorem 1.1. *Assume (1.6)-(1.7)-(1.8). For any $\varepsilon \geq 0$, there exists at least one solution $(F_\varepsilon, M_\varepsilon) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ to the stationary problem (1.5) such that*

$$0 \leq F_\varepsilon(x) \leq C e^{-\frac{a_0}{2}x}, \quad |F'_\varepsilon(x)| \leq C e^{-\frac{a_0}{2}x}, \quad \forall x \geq 0, \quad (1.10)$$

for a constant $C \in (0, \infty)$. Moreover, there exists $\varepsilon_0 > 0$, small enough, such that the above solution is unique for any $\varepsilon \in [0, \varepsilon_0)$.

For a given initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+)$, we say that a function f is a weak (positive and mass conserving) solution to (1.1)–(1.3) if

$$0 \leq f \in C([0, \infty); L^1(\mathbb{R})), \quad \langle f(t) \rangle = \langle f_0 \rangle, \quad \forall t \geq 0,$$

and f satisfies (1.1) in the distributional sense $\mathcal{D}'([0, \infty) \times [0, \infty))$ for some functions $m, p \in C([0, \infty))$ which fulfilled the constraints (1.2) and (1.3).

Under the above assumptions, existence and uniqueness of weak solutions have been established in [26, Theorem 1.1]. The main concern of the present work is the following long-time asymptotic result on the solutions.

Theorem 1.2. *We assume that the firing rate a satisfies (1.6), (1.7) and (1.8). We also assume that the delay distribution b satisfies $b = \delta_0$ or (1.9). There exists $\varepsilon_0 > 0$, small enough, and there exist some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ such that for any connectivity parameter $\varepsilon \in (0, \varepsilon_0)$ and any initial datum $0 \leq f_0 \in L^1$ with mass 1 and such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$, the unique solution f to the evolution equation (1.1)–(1.3) satisfies*

$$\|f(t, \cdot) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \quad \forall t \geq 0.$$

In other words, in that weak connectivity regime, we prove that the total asynchronous firing of neurons appears exponentially fast in the large time asymptotic. Theorem 1.2 extends to firing rates a satisfying (1.6)–(1.8) some similar exponential stability results obtained in [16, 17] in the case without delay and for a *step function* firing rate a given by

$$a(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \leq 0. \quad (1.11)$$

It is worth mentioning that the above firing rate does not fall in the class of rates considered in the present paper because condition (1.8) is not met. On

the other hand, we are able to tackle the case without and with delay in the same time, what it was not the case in [16, 17]. In the delay case, stability results were established in [16], but not exponential stability.

Our proof follows a strategy of “perturbation of semigroup” initiated in [12] for studying long time convergence to the equilibrium for the homogeneous inelastic Boltzmann equation and used recently in [13] for a neuron network equation based on a brownian (hypoelliptic) perturbation of the well-known FitzHugh-Nagumo dynamic. More precisely, we introduce the linearized equation for the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$, which writes

$$\partial_t g = -\partial_x g - a(x, \varepsilon M_\varepsilon)g - n(t) \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon, \quad (1.12a)$$

$$g(t, 0) = q(t), \quad g(0, x) = g_0(x), \quad (1.12b)$$

with

$$q(t) = \int_0^\infty a(x, \varepsilon M_\varepsilon) g \, dx + n(t) \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \, dx \quad (1.13)$$

and

$$n(t) := \int_0^\infty q(t - y) b(dy). \quad (1.14)$$

We associate to that linear evolution equation a generator Λ_ε (which acts on an appropriate space to be specified in the two cases without and with delay) and its semigroup S_{Λ_ε} . It turns out that we may split the operator Λ_ε as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

for some α -hypodissipative operator \mathcal{B}_ε , $\alpha < 0$, and some bounded and \mathcal{B}_ε -power regular operator \mathcal{A}_ε as defined in [24, 8, 14, 10, 11]. In particular, adapted versions of the Spectral Mapping Theorem in [14, 10, 11] and the Weyl’s Theorem in [24, 8, 14, 10, 11] imply that the semigroup S_{Λ_ε} has a finite dimensional dominant part. Moreover, in the limit case when $\varepsilon = 0$, the term $n(t)$ disappears from equation (1.12) and the resulting semigroup S_{Λ_0} becomes positive. That allows us to use the Krein-Rutman Theorem established in [14, 10, 11] in order to get that the stationary state (F_0, M_0, M_0) is unique and exponentially stable. Using next a perturbative argument developed in [12, 23, 11, 15], we get that the unique stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ is also exponentially stable in the weak connectivity regime. We conclude the

proof of Theorem 1.2 by a somewhat classical nonlinear exponential stability argument.

The same strategy applies to the case without delay and with delay. In both cases, the boundary condition in the age structure equation is treated as a source term and, in the delay case, the delay equation (1.4) is replaced by a simple age equation on an auxiliary function, so that the resulting linearized equation writes as an autonomous system of two PDEs and falls in the classical framework of linear evolution equations generating a semigroup.

Our approach is thus quite different from the usual way to deal with delay equations, as introduced by I. Fredholm [6] and V. Volterra [25], which consists in using the specific framework of “fading memory space”, which goes back at least to Coleman & Mizel [1], or the theory of “abstract algebraic-delay differential systems” developed by O. Diekmann and co-authors [3].

Our approach is also different from the previous works [16, 17, 18] where the asymptotic stability analysis were performed by taking advantage of the step function structure (1.11) of the firing rate. That one makes possible to explicitly exhibit a suitable norm (related to the W_1 Monge-Kantorovich-Wasserstein optimal transport distance) such that some related linear age structure operator is dissipative. The present method is based on a more abstract approach but in the other hand it is somewhat more flexible because it does not require to explicitly exhibit a norm for which the underlying linear(ized) operators are dissipative. In particular, we hope that our strategy can be adapted to the large connectivity regime, to the step function firing rate (1.11) as well as to models including fragmentation term to describe neuronal networks with adaptation and fatigue, and thus generalize to the case with a delay term all the stability results established in [16, 17, 18] in the case without delay.

Let us end the introduction by describing the plan of the paper. In Section 2, we introduce the strategy, we prove the stationary state result and we establish Theorem 1.2 in the case without delay. In Section 3, we establish Theorem 1.2 in the case with delay. As we mentioned above, the strategy of proof for the case without and with delay is rather the same. For pedagogical reason, we start presenting the method on the simplest “*without delay case*” in Section 2, where we prove the stationary problem result Theorem 1.1 as well as Theorem 1.2 in that case. Next, in Section 3, we only explain how the proof must be modified in order to treat the more complicated “*with delay case*” and thus establish Theorem 1.2 in all generality.

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2 Case without delay

The present section is devoted to the proof of our main result Theorem 1.2 in the without delay case.

2.1 The stationary problem

We first deal with the stationary problem and we prove the existence of steady state as well as its uniqueness in the small connectivity regime.

Proof of Theorem 1.1. Step 1. We prove the existence of a solution. We set

$$A(x, m) := \int_0^x a(y, m) dy, \quad \forall x, m \geq 0.$$

For any $m \geq 0$, we can solve the equation (1.5a), by writing

$$F_{\varepsilon, m}(x) := T_m e^{-A(x, \varepsilon m)}, \quad (2.1)$$

where $T_m \geq 0$ is chosen in order that $F_{\varepsilon, m}$ satisfies the mass normalized condition, namely

$$T_m^{-1} = \int_0^\infty e^{-A(x, \varepsilon m)} dx.$$

In order to conclude the existence of a solution, we just have to find a real number $m = M_\varepsilon$ such that $m = F_{\varepsilon, m}(0) = T_m$. Equivalently, we need to find $M_\varepsilon \geq 0$ such that

$$\Phi(\varepsilon, M_\varepsilon) = 1, \quad (2.2)$$

where

$$\Phi(\varepsilon, m) = m T_m^{-1} := m \int_0^\infty e^{-A(x, \varepsilon m)} dx.$$

From the assumption (1.7) of a , there exists $x_0 \in [0, \infty)$ such that $a(x, \mu) \geq \frac{a_0}{2}$, for any $x \geq x_0$, $\mu \geq 0$, and therefore

$$\frac{a_0}{2}(x - x_0)_+ \leq A(x, \mu) \leq a_1 x, \quad \forall x \geq 0, \quad \forall \mu \geq 0. \quad (2.3)$$

Thanks to the Lebesgue dominated convergence theorem, we deduce that $\Phi(\varepsilon, \cdot)$ is a continuous function. Because $\Phi(\varepsilon, 0) = 0$, $\Phi(\varepsilon, \infty) = \infty$ and thanks to the intermediate value theorem, we conclude to the existence of at least one real number $M_\varepsilon \in (0, \infty)$ such that (2.2) holds. Estimates (1.10) immediately follow from the identity (2.1) and the estimate (2.3).

Step 2. We prove the uniqueness of the solution in the weak connectivity regime. Obviously, there exists a unique $M_0 := (\int_0^\infty e^{-A(x,0)} dx)^{-1} \in (0, \infty)$ such that $\Phi(0, M_0) = 1$. Moreover, we compute

$$\frac{\partial}{\partial m} \Phi(\varepsilon, m) = \int_0^\infty e^{-A(x, \varepsilon m)} (1 - m\varepsilon \frac{\partial A}{\partial m}(x, \varepsilon m)) dx,$$

which is continuous as a function of the two variables because of (1.8). We then easily obtain that $\Phi \in C^1$. Since moreover

$$\frac{\partial}{\partial m} \Phi(\varepsilon, m)|_{\varepsilon=0} = \int_0^\infty e^{-A(x,0)} dx > 0,$$

the implicit function theorem implies that there exists $\varepsilon_0 > 0$, small enough, such that the equation (2.2) has a unique solution for any $\varepsilon \in [0, \varepsilon_0]$. \square

Remark 2.1. *In the above proof, we do not need (1.8) but only the weaker smoothness assumption that A and $\partial_m A$ are continuous.*

2.2 Linearized equation and structure of the spectrum

To go one step further, we introduce the linearized equation around the stationary solution $(F_\varepsilon, M_\varepsilon)$. On the variation (g, n) , the linearized equation writes

$$\begin{aligned} \partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t) &= 0, \\ g(t, 0) = n(t) &= \int_0^\infty (a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t)) dx, \quad g(0, x) = g_0(x), \end{aligned}$$

with $a_\varepsilon := a(x, \varepsilon M_\varepsilon)$, $a'_\varepsilon := \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon)$. Since there exists $\varepsilon_0 > 0$, small enough, such that

$$\forall \varepsilon \in (0, \varepsilon_0) \quad \kappa := \int_0^\infty a'_\varepsilon F_\varepsilon dx < 1,$$

we may define

$$\mathcal{M}_\varepsilon[g] := (1 - \kappa)^{-1} \int_0^\infty a_\varepsilon g \, dx, \quad (2.4)$$

and the linearized equation is then equivalent to

$$\partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g(t, \cdot)] = 0, \quad (2.5)$$

$$g(t, 0) = \mathcal{M}_\varepsilon[g(t, \cdot)], \quad g(0, x) = g_0(x). \quad (2.6)$$

To the above linear evolution equation, one can classically associate a semigroup $S_{L_\varepsilon}(t)$ acting on $X := L^1(\mathbb{R}_+)$, with generator

$$L_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g]$$

and domain

$$D(L_\varepsilon) := \{g \in W^{1,1}(\mathbb{R}_+); \, g(0) = \mathcal{M}_\varepsilon[g]\}.$$

Here we also use another approach by considering the boundary term as a source term, and then rewriting the equation as

$$\partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g] + \delta_{x=0} \mathcal{M}_\varepsilon[g], \quad (2.7)$$

acting on the space of bounded Radon measures

$$\mathcal{X} := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \, \text{supp } g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$. Here and below, $C_0(I)$ denotes the space of continuous functions on the closed interval $I = \mathbb{R}$ or $I = \mathbb{R}_+$ which goes to 0 at infinity. We then denote by $S_{\Lambda_\varepsilon}(t)$ the semigroup on \mathcal{X} generated by Λ_ε .

It is worth emphasizing that for any $g_0 \in X$, the function $g(t) = S_{L_\varepsilon}(t)g_0 \in C([0, \infty); X)$ is a weak solution to equations (2.5)-(2.6), and more precisely clearly satisfies

$$-\int_0^\infty \varphi(0) g_0 \, dx + \int_0^\infty \int_0^\infty g \{ -\partial_t \varphi + \Lambda_\varepsilon^* \varphi \} \, dx dt = 0,$$

for any $\varphi \in C_c^1([0, \infty); C_0(\mathbb{R}_+)) \cap C_c([0, \infty); C_0^1(\mathbb{R}_+))$. Here, we have defined

$$\Lambda_\varepsilon^* \psi := \partial_x \psi - a_\varepsilon \psi - a_\varepsilon (1 - \kappa)^{-1} \left[\int_0^\infty a'_\varepsilon(y) F_\varepsilon(y) \psi(y) \, dy - \psi(0) \right],$$

for any $\psi \in C_0^1(\mathbb{R}_+)$, the space of C^1 functions which goes to 0 at infinity as well as their first derivative. As a consequence, the semigroup S_{Λ_ε} being defined by duality from the semigroup $S_{\Lambda_\varepsilon^*}$, we have $S_{\Lambda_\varepsilon}|_X = S_{L_\varepsilon}$.

For a generator L , we denote by $\Sigma(L)$ its spectrum and by S_L the associated semigroup. We refer to the classical textbooks [9, 19, 4] for an introduction to the spectral analysis of operators and the semigroup theory. Our next result deals with the structure of the spectral set $\Sigma(\Lambda_\varepsilon)$ of Λ_ε and the splitting structure of the associated semigroup S_{Λ_ε} .

Theorem 2.2. *Assume (1.6)-(1.7)-(1.8) and define $a^* := -a_0/2 < 0$. The operator Λ_ε is the generator of a weakly $*$ continuous semigroup S_{Λ_ε} acting on \mathcal{X} endowed with the weak $*$ topology $\sigma(M^1, C_0)$. Moreover, there exists a finite rank projector $\Pi_{\Lambda_\varepsilon, a^*}$ which commutes with S_{Λ_ε} , an integer $j \geq 0$ and some complex numbers*

$$\xi_1, \dots, \xi_j \in \Delta_{a^*} := \{z \in \mathbb{C}, \Re z > a^*\},$$

such that on $E_1 := \Pi_{\Lambda_\varepsilon, a^*} \mathcal{X}$ the restricted operator satisfies

$$\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_{a^*} = \{\xi_1, \dots, \xi_j\} \quad (2.8)$$

(with the convention $\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_{a^*} = \emptyset$ when $j = 0$) and for any $a > a^*$ there exists a constant C_a such that the remainder semigroup satisfies

$$\|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon, a^*})\|_{\mathcal{B}(\mathcal{X})} \leq C_a e^{at}, \quad \forall t \geq 0. \quad (2.9)$$

The proof of the result is a direct consequence of the fact that the operator Λ_ε splits as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ where \mathcal{A}_ε and \mathcal{B}_ε are defined on \mathcal{X} by

$$\mathcal{A}_\varepsilon g := \gamma_\varepsilon \mathcal{M}_\varepsilon[g], \quad \gamma_\varepsilon := \delta_0 - a'_\varepsilon F_\varepsilon, \quad (2.10)$$

$$\mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g, \quad (2.11)$$

for which we can adapt Weyl's Theorem of [24, 8, 14, 10, 11] and the Spectral Mapping Theorem of [14, 10, 11]. The picture may seem not that simple, because Λ_ε does not generate a strongly continuous semigroup on \mathcal{X} and then apparently does not fit the framework developed in [14, 10]. However, we may probably circumvent that issue in the following ways:

- We may observe that the strong continuity is little used in [14, 10] and then the results stated therein extend to weakly $*$ continuous semigroups. In other words, on the finite dimensional eigenspace associated to the principal

part of the spectrum (2.8) continuity and weak $*$ continuity are equivalent, while in the remainder part (2.9) we just use a decay bound which does not require the strong continuity property, see [19, Chapter 1] as well as [26, 11] where such a weak $*$ continuous framework is also discussed.

- We may apply the theory developed in [14, 10] to the adjoint operator Λ_ε^* and the associated semigroup acting on $C_0(\mathbb{R})$ and then deduce the result by duality.

- We may probably reduce the space X by “sun duality”, see [4, Chapter II.2.6], apply the theory developed in [14, 10] to the resulting strongly continuous semigroup and then conclude by a density argument.

We rather follow another strategy, which has the advantage to be more self-contained and pedagogical, by adapting to our context the proofs from [14, 10]. It is worth emphasizing that we introduce the two spaces $X \subset \mathcal{X}$ because the splitting $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ has some clearer meaning in the space \mathcal{X} while the Banach structure of X is used when we establish the adequate version of Weyl’s theorem.

We recall the definition of hypodissipativity introduced in [8]. We say that the closed operator L on a Banach space E with dense domain $D(L)$ is α -hypodissipative if there exists an equivalent norm $||| \cdot |||$ on E such that

$$\forall f \in D(L), \exists \varphi \in F_{|||, |||}(f) \quad \Re \langle \varphi, (L - \alpha) f \rangle \leq 0,$$

where, for any $f \in E$, the associated dual set $F_{|||, |||}(f) \subset E'$ is defined by

$$F_{|||, |||}(f) := \{ \varphi \in E'; \langle \varphi, f \rangle = |||f|||_X^2 = |||\varphi|||_{E'}^2 \}.$$

We also recall that the generator L of a semigroup of bounded operators is α -hypodissipative if, and only if, there exists a constant $M \geq 1$ such that the associated semigroup S_L on E satisfies the growth estimate

$$\|S_L(t)\|_{\mathcal{B}(E)} \leq M e^{\alpha t}, \quad \forall t \geq 0,$$

where $\mathcal{B}(E)$ denotes the space of linear and bounded operators on E . We will sometime abuse by saying that S_L is α -hypodissipative when it satisfies the above growth estimate. We refer to [19, 8, 14, 11] for details.

We start with the properties of the two auxiliary operators.

Lemma 2.3. *Assume that \mathcal{A} satisfies conditions (1.6)–(1.8). For any $\varepsilon \geq 0$, the operators \mathcal{A}_ε and \mathcal{B}_ε satisfy :*

(i) $\mathcal{A}_\varepsilon \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathcal{X})$.

(ii) $S_{\mathcal{B}_\varepsilon}$ is a^* -hypodissipative in both X and \mathcal{X} .

(iii) The family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{B}(\mathcal{X}, Y)} \leq C_a e^{at}, \quad \forall a > a^*,$$

for some constant $C_a \in (0, \infty)$ and with $Y := BV(\mathbb{R}_+) \cap L_1^1(\mathbb{R}_+)$.

Here $BV(\mathbb{R}_+)$ stands for the space of functions with bounded variation and $L_1^1(\mathbb{R}_+)$ stands for the weighted Lebesgue space associated to the weight function $x \mapsto \langle x \rangle$.

Proof. In order to shorten notation we skip the ε dependency, we write $a(x) = a(x, \varepsilon M_\varepsilon)$, $A(x) = A(x, \varepsilon M_\varepsilon)$, $S_\Lambda = S_{\Lambda_\varepsilon}$, $S_\mathcal{A} = S_{\mathcal{A}_\varepsilon}$, $S_\mathcal{B} = S_{\mathcal{B}_\varepsilon}$ and so on.

Step 1. Proof of (i). We have $\mathcal{A} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathcal{X})$ from the fact that $\mathcal{M}_\varepsilon[\cdot] \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathbb{R})$ because $\|a\|_{W^{1,\infty}} < \infty$ as assumed in (1.8).

Step 2. Proof of (ii). We use the same notation \mathcal{B} and $S_\mathcal{B}$ for the generator and its associated semigroup defined in the spaces X and \mathcal{X} . We write $S_\mathcal{B}$ with the explicit formula

$$S_\mathcal{B}(t)g(x) = e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0} =: S(t, x). \quad (2.12)$$

From the inequality $a(z, 0) \geq (3a_0/4)(1 - \mathbf{1}_{0 \leq z \leq x_1})$ for some $x_1 \in (0, \infty)$ coming from (1.7), we deduce $A(x-t) - A(x) \leq 3a_0x_1/4 - (3a_0/4)t$ for any $x \geq t \geq 0$, next

$$e^{A(x-t)-A(x)} \leq C e^{3\beta t}, \quad \forall x \geq t \geq 0, \quad (2.13)$$

with $C := e^{\frac{3a_0x_1}{4}} > 0$, $\beta := -a_0/4 < 0$, and finally

$$\|S_\mathcal{B}(t)g\|_X \leq C e^{3\beta t} \|g\|_X, \quad \forall t \geq 0, \forall g \in X.$$

We conclude by observing that $3\beta < a^*$ and that the same estimate holds with X replaced by \mathcal{X} thanks to a (weakly $*$) density argument.

Step 3. Proof of (iii). For $g \in \mathcal{X}$ and with the notation of Step (ii), we have

$$\mathcal{A}S_\mathcal{B}(t)g = \gamma_\varepsilon N(t),$$

where γ_ε is defined in (2.10) and

$$N(t) := \mathcal{M}_\varepsilon[S(t, \cdot)] = (1 - \kappa)^{-1} \int_0^\infty a(x) e^{A(x-t)-A(x)} g(x-t) \mathbf{1}_{x-t \geq 0} dx.$$

It is worth noticing that $N \in C_b(\mathbb{R}_+)$, because $(x, t) \mapsto a(x) e^{A(x-t)-A(x)}$ is a bounded and continuous function on the set $\mathbb{T} := \{(x, t); x \geq t \geq 0\}$. Moreover, as in Step (ii), we have

$$|N(t)| \leq C a_1 \int_0^\infty e^{3\beta t} |g(x-t)| \mathbf{1}_{x-t \geq 0} dx \leq C a_1 e^{3\beta t} \|g\|_{\mathcal{X}}, \quad (2.14)$$

for any $t \geq 0$. We deduce

$$\begin{aligned} (S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g(x) &= \int_0^t (S_{\mathcal{B}}(s)\gamma_\varepsilon)(x) N(t-s) ds \\ &= \int_0^t e^{A(x-s)-A(x)} \gamma_\varepsilon(x-s) N(t-s) \mathbf{1}_{x-s \geq 0} ds \quad (2.15) \\ &= e^{-A(x)} (\nu_\varepsilon * \check{N}_t)(x), \end{aligned}$$

$$(2.16)$$

with $\nu_\varepsilon := \gamma_\varepsilon e^A$ and the classical notation $\check{N}_t(s) = N(t-s)$. Starting from identity (2.15) and denoting $e_{-\beta}(x) := e^{-\beta x}$, we have

$$\begin{aligned} |(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g(x)| e_{-\beta}(x) &\leq e^{-\beta x} \int_0^t C e^{3\beta t} |\gamma_\varepsilon(x-s)| |N(t-s)| ds \\ &= C e^{a^* t} \int_0^t e^{2\beta(t-s)} \{e^{-\beta(x-s)} |\gamma_\varepsilon(x-s)|\} \{e^{-\beta(t-s)} |N(t-s)|\} ds \\ &\leq C e^{a^* t} \|(|\gamma_\varepsilon| e_{-\beta}) * (|N| e_{-\beta})\|_{C_b} \\ &\leq C e^{a^* t} \|\gamma_\varepsilon e_{-\beta}\|_{\mathcal{X}} \|N e_{-\beta}\|_{C_b}. \end{aligned}$$

Using the definition (2.10) and the estimates (1.10) and (2.14), we deduce

$$\|(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g e_{-\beta}\|_{L^\infty} \leq C' e^{a^* t} \|g\|_{\mathcal{X}}, \quad \forall t \geq 0, \quad (2.17)$$

for some constant $C' \in (0, \infty)$.

Next, differentiating the functions in both sides of identity (2.16), we get

$$\begin{aligned} \partial_x[(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g](x) &= -a(x) e^{-A(x)} (\nu_\varepsilon * \check{N}_t)(x) - e^{-A(x)} (\nu_\varepsilon * \check{N}'_t)(x) \\ &\quad - e^{-A(x)} \nu_\varepsilon(x-t) N(0) \mathbf{1}_{x-t \geq 0} + e^{-A(x)} \nu_\varepsilon(x) N(t), \end{aligned}$$

with $\check{N}'_t(s) = N'(t-s)$. In order to estimate the second term, we compute

$$\begin{aligned} N'(t) &= (1-\kappa)^{-1} \int_0^\infty \partial_x [a(x)e^{-A(x)}] e^{A(x-t)} g(x-t) \mathbf{1}_{x-t \geq 0} dx \\ &= (1-\kappa)^{-1} \int_0^\infty [a'(x) - a(x)^2] e^{A(x-t)-A(x)} g(x-t) \mathbf{1}_{x-t \geq 0} dx. \end{aligned}$$

Using the inequality (2.13), we deduce

$$\begin{aligned} |N'(t)| &\leq C' \int_0^\infty \{|a'(x)| + a(x)^2\} e^{3\beta t} g(x-t) \mathbf{1}_{x-t \geq 0} dx dt \\ &\lesssim e^{a^* t} \|g\|_{\mathcal{X}}. \end{aligned}$$

As a consequence, using again (2.13), we get

$$\begin{aligned} \|e^{-A}(\nu_\varepsilon * \check{N}'_t)\|_{\mathcal{X}} &\lesssim \int_0^t \int_0^\infty e^{3\beta s} |\gamma_\varepsilon(x-s)| |N'(t-s)| dx ds \\ &\lesssim e^{a^* t} \|\gamma_\varepsilon\|_{\mathcal{X}} \|N'\|_{\mathcal{X}} \lesssim e^{a^* t} \|g\|_{\mathcal{X}}. \end{aligned}$$

Treating in a similar way the two other terms involved in the expression of $\partial_x[(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g]$, we finally obtain

$$\|\partial_x[(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)g]\|_{\mathcal{X}} \lesssim e^{a^* t} \|g\|_{\mathcal{X}}.$$

Together with (2.17) that concludes the proof of (iii). \square

Proof of Theorem 2.2. The main idea is to apply or adapt the versions of Weyl's Theorem [14, Theorem 3.1], [10, Theorem E.3.1] and of the Spectral Mapping Theorem [14, Theorem 2.1], [10, Theorem E.2.1] (see also the variant results in [11]). We start collecting the three key properties satisfied by the operators \mathcal{A}_ε and \mathcal{B}_ε involved in the splitting of Λ_ε

We denote by X_ζ the abstract Sobolev space associated to L_ε for $\zeta \in \mathbb{R}$, see e.g. [4, Section II.5], so that in particular $X_\zeta \subset W^{\zeta,1}(\mathbb{R}_+)$, with equality when $\zeta \leq 0$. We recall that for the generator L of a semigroup S_L , we define the resolvent set $\rho(L)$ by

$$\rho(L) := \{z \in \mathbb{C}; L - z \text{ is a bijection}\} = \mathbb{C} \setminus \Sigma(L),$$

as well as the resolvent (operator) $R_L(z) := (L - z)^{-1}$ for any $z \in \rho(L)$. We finally define the half plane $\Delta_a := \{z \in \mathbb{C}; \Re z > a\}$ for any $a \in \mathbb{R}$.

- (A1) For any $a > a^*$ and $\ell \in \mathbb{N}$, there exists a positive constant $C_{a,\ell}$ such that the following growth estimate holds

$$\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{\mathcal{B}(X)} \leq C_{a,\ell} e^{at}, \quad \forall t \geq 0. \quad (2.18)$$

It is obvious that (A1) is true for $\ell = 0$ from Lemma 2.3–(ii) and for $\ell = 1$ from (2.17). We then deduce (A1) for any $\ell \geq 2$ and $a > a^*$ by writing $S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)} = [S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})] * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell-1)}$ and by using that $(\mathcal{B}_\varepsilon - a^*)$ is hypodissipative in \mathcal{X} and $\mathcal{A}_\varepsilon \in \mathcal{B}(\mathcal{X})$.

- (A2) There holds $\mathcal{A}_\varepsilon \in \mathcal{B}(X_{-1}, X_\zeta)$ for any $\zeta \in (-1, 0)$ and the family of operators $(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*2)}(t)$ satisfies the estimate

$$\|(S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*2)}(t)\|_{\mathcal{B}(X_{-1}, X)} \leq C'_{a^*,1} e^{a^* t}, \quad \forall t \geq 0, \quad (2.19)$$

for a positive constant $C'_{a,1}$. The first claim is a consequence of the continuous embedding $\mathcal{X} \subset W^{\zeta,1}$ and of Lemma 2.3–(i). The second estimate is obtained by putting together the properties (i) and (ii) in Lemma 2.3.

- (A3) The family of operators $(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^3(z)$ satisfies the compactness estimate

$$\|(R_{\mathcal{B}_\varepsilon}(z) \mathcal{A}_\varepsilon)^3\|_{\mathcal{B}(\mathcal{X}, Y)} \leq C''_{2,a}, \quad \forall z \in \Delta_a, \quad \forall a > a^*,$$

for some positive constant $C''_{2,a}$. Observing that $z \mapsto -(R_{\mathcal{B}_\varepsilon}(z) \mathcal{A}_\varepsilon)^3$ is nothing but the Laplace transform of the function $t \mapsto (S_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^{(*3)}(t)$, that clearly holds true thanks to Lemma 2.3.

We now briefly explain how the proof goes on, referring to [14, 10, 11] for more details and developments. In order to shorten notation we skip again the ε dependency.

Step 1. On the one hand, because of (2.19) and the fact that $R_{\mathcal{B}}$ is nothing but the opposite of the Laplace transform of $S_{\mathcal{B}}$, we have

$$\|(R_{\mathcal{B}}(z) \mathcal{A})^2\|_{X_{-1} \rightarrow X} \leq C, \quad \forall z \in \Delta_a. \quad (2.20)$$

Recalling that the negative abstract Sobolev norm is defined by

$$\|g\|_{X_{-1}} := \|R_{\mathcal{B}}(a)g\|_X,$$

and using the dissipativity property of \mathcal{B} , we immediately have

$$\forall z \in \Delta_a, \forall g \in X, \quad \|R_{\mathcal{B}}(z)g\|_{X_{-1}} \leq C \|g\|_{X_{-1}}.$$

Moreover, using the resolvent identity

$$\begin{aligned} R_{\mathcal{B}}(a)R_{\mathcal{B}}(z) &= z^{-1} (R_{\mathcal{B}}(a)\mathcal{B}R_{\mathcal{B}}(z) - R_{\mathcal{B}}(a)), \\ &= z^{-1} (R_{\mathcal{B}}(z) - R_{\mathcal{B}}(a) + aR_{\mathcal{B}}(z)R_{\mathcal{B}}(a)), \end{aligned}$$

we also have

$$\begin{aligned} \|R_{\mathcal{B}}(z)g\|_{X_{-1}} &\leq \frac{1}{|z|} \{ \|R_{\mathcal{B}}(z)g\|_X + \|R_{\mathcal{B}}(a)g\|_X + |a| \|R_{\mathcal{B}}(z)R_{\mathcal{B}}(a)g\|_X \} \\ &\leq \frac{C}{|z|} \|g\|_X, \end{aligned}$$

for any $g \in X$ and $z \in \Delta_a$. Finally, using both estimates and an interpolation argument, we deduce

$$\forall z \in \Delta_a, \quad \|R_{\mathcal{B}}(z)\|_{X_{\zeta} \rightarrow X_{-1}} \leq \frac{C}{\langle z \rangle^{1+\zeta}}, \quad \forall \zeta \in (-1, 0). \quad (2.21)$$

We conclude with

$$\forall z \in \Delta_a, \quad \|(R_{\mathcal{B}}(z)\mathcal{A})^3\|_{\mathcal{X} \rightarrow X} \leq \frac{C}{\langle z \rangle^{3/4}}, \quad (2.22)$$

by writing

$$(R_{\mathcal{B}}(z)\mathcal{A})^3 = (R_{\mathcal{B}}(z)\mathcal{A})^2 R_{\mathcal{B}}(z)\mathcal{A}$$

and by using estimates (i) in Lemma 2.3 with $\zeta = -1/4$, (2.21) with $\zeta = -1/4$ and (2.20).

Step 2. Here, we follow [14, Proof of Theorem 2.1] and [24] by taking advantage of the analysis of degenerate-meromorphic functions performed in [21]. Iterating the factorization identity $R_{\Lambda} = R_{\mathcal{B}} - R_{\mathcal{B}}\mathcal{A}R_{\Lambda}$, we obtain

$$(I - \mathcal{V})R_{\Lambda} = \mathcal{U} \quad \text{in} \quad \mathcal{B}(\mathcal{X}) \quad (2.23)$$

with

$$\mathcal{U} = R_{\mathcal{B}} - R_{\mathcal{B}}\mathcal{A}R_{\mathcal{B}} + (R_{\mathcal{B}}\mathcal{A})^2 R_{\mathcal{B}}, \quad \mathcal{V} := -(R_{\mathcal{B}}\mathcal{A})^3.$$

Because of (2.22), (2.18) and of the identity

$$R_{\mathcal{B}}(\mathcal{A}R_{\mathcal{B}})^{\ell}(z) = (-1)^{\ell+1} \int_0^{\infty} e^{-zt} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t) dt \in \mathcal{B}(X)$$

for any $z \in \Delta_a$, we see that all the terms involved in (2.23) act in $\mathcal{B}(X)$, so that we also have

$$(I - \mathcal{V})R_L = \mathcal{U} \quad \text{in} \quad \mathcal{B}(X). \quad (2.24)$$

Because of (2.22), we see that $I - \mathcal{V}$ is invertible on $\Delta_a \setminus B(0, M)$ for $M > 0$ large enough and (2.24) tells us that $R_L = (I - \mathcal{V})^{-1}\mathcal{U}$ is uniformly bounded in $\Delta_a \setminus B(0, M)$, in particular $\Sigma(L) \subset \Delta_a^c \cup B(0, M)$.

On the other hand, because of **(A3)**, there holds

$$\|\mathcal{V}(z)\|_{X \rightarrow Y} \leq C,$$

with compact embedding $Y \subset X$. As a consequence of the fact that \mathcal{V} is the opposite of the Laplace transform of the nice function $\mathbb{R}_+ \rightarrow X$, $t \mapsto (S_{\mathcal{B}}\mathcal{A})^{(*3)}(t)$, we also deduce that \mathcal{V} is holomorphic in $\mathcal{B}(X)$ on Δ_a . We define $\Psi(z) := I - \mathcal{V}(z)$. Because Ψ is an holomorphic function on Δ_a , $\Psi(z)$ is invertible for (some) $z \in \Delta_a \setminus B(0, M)$ and $R\mathcal{V}(z) \subset Y$ for any $z \in \Delta_a$, [21, Corollary II] asserts that the function $z \mapsto \Psi(z)^{-1}$ is a degenerate-meromorphic operators valued function, that is Ψ is holomorphic on $\Delta_a \setminus D$, $D \subset \Delta_a$ is discrete, any point $\xi \in D$ is an isolate pole and the coefficients of the principal part in the Laurent series are finite rank operators. Next, from [21] and the identity

$$R_L = \Psi^{-1}\mathcal{U},$$

we deduce that R_L is also degenerate-meromorphic on Δ_a , in particular $\Sigma(L) \cap \Delta_a$ is a finite set of discrete eigenvalues, which means isolated eigenvalues associated to an algebraically finite eigenspace. That is nothing but (2.8). We define $\Pi := I - \Pi_{\Lambda, a}$.

Step 3. We claim that for any $n \geq 2$, there holds

$$\begin{aligned} S_L(t)\Pi &= \{S_{\mathcal{B}}(t) + \dots + (S_{\mathcal{B}}\mathcal{A})^{*(n-1)} * S_{\mathcal{B}}(t)\}\Pi \\ &\quad + (-1)^{n+1} \frac{i}{2\pi} \int_{\uparrow_a} (R_{\mathcal{B}}\mathcal{A})^n(z) R_L(z)\Pi e^{tz} dz \end{aligned} \quad (2.25)$$

in $\mathcal{B}(X)$ where $\uparrow_c := \{c + iy, y \in \mathbb{R}\}$ is the complex line of abscissa $c \in \mathbb{R}$. In order to prove that identity, we iterate the Duhamel formula

$$S_{\Lambda}(t) = S_{\mathcal{B}}(t) + S_{\mathcal{B}}\mathcal{A} * S_{\Lambda}(t),$$

in $\mathcal{B}(\mathcal{X})$ and for any $f \in X$ and $n \geq 2$. We then obtain

$$S_L(t)\Pi f = \{S_{\mathcal{B}}(t) + \dots + (S_{\mathcal{B}}\mathcal{A})^{*(n-1)} * S_{\mathcal{B}}(t)\}\Pi f + g_n(t),$$

with

$$g_n(t) := (S_{\mathcal{B}}\mathcal{A})^{*(n)} * S_L(t)\Pi f.$$

Because $t \mapsto g'_n(t)e^{-bt} = \mathcal{B}g_n(t)e^{-bt} + \mathcal{A}g_{n-1}(t)e^{-bt} \in L^1(\mathbb{R}_+, \mathcal{X})$ for $b > 0$ large enough from **(A1)**-**(A2)** and the fact that $t \mapsto S_L(t)\Pi f e^{-bt} \in L^\infty(X)$, the following inverse Laplace transform formula holds

$$g_n(t) = (-1)^{n+1} \frac{i}{2\pi} \int_{\uparrow_b} (R_{\mathcal{B}}\mathcal{A})^n(z) R_L(z)\Pi f e^{tz} dz, \quad \forall t \geq 0.$$

From the preceding steps, the function under the integral sign is holomorphic on Δ_{a^*} and we may move the line of integration from \uparrow_b to \uparrow_a . Observing that every term in the resulting identity belongs to X , we conclude to (2.25). More precisely and importantly, taking $n = 6$, using (2.22) and the fact that $\|R_L(z)\Pi\|_{\mathcal{B}(X)}$ is uniformly bounded on Δ_a from Step 2, we have

$$\|g_n(t)\|_X \leq \frac{e^{at}}{2\pi} \int_{\uparrow_a} \|(R_{\mathcal{B}}\mathcal{A})^6(z)\|_{\mathcal{B}(X)} \|R_L(z)\Pi\|_{\mathcal{B}(X)} dz \|f\|_X \lesssim e^{at} \|f\|_X,$$

for any $t \geq 0$. We conclude to (2.9) using that the other terms involved in (2.25) are similarly bounded thanks to **(A1)** and using a weakly $*$ density argument. \square

2.3 The vanishing connectivity regime

When the network connectivity parameter vanishes, $\varepsilon = 0$, the linearized time elapsed operator simplifies into

$$\Lambda_0 g = -\partial_x g - a(x, 0)g + \delta_{x=0} \mathcal{M}_0[g], \quad (2.26)$$

where $\mathcal{M}_0[g] = \int_0^\infty a(x, 0)g(x)dx$. The associated semigroup is then positive and the following version of the Krein-Rutman theorem holds.

Theorem 2.4. *There exist some constants $\alpha_0 < 0$ and $C > 0$ such that $\Sigma(\Lambda_0) \cap \Delta_{\alpha_0} = \{0\}$ and for any $g_0 \in \mathcal{X}$, $\langle g_0 \rangle = 0$, there holds*

$$\|S_{\Lambda_0}(t)g_0\|_{\mathcal{X}} \leq Ce^{\alpha_0 t} \|g_0\|_{\mathcal{X}}, \quad \forall t \geq 0. \quad (2.27)$$

We denote

$$\mathcal{X}_+ := \{g \in M^1(\mathbb{R}_+); g \geq 0\},$$

the space of bounded and nonnegative Radon measures.

We start with two elementary auxiliary results.

Lemma 2.5. *S_{Λ_0} is positive: $S_{\Lambda_0}(t)g \in \mathcal{X}_+$ for any $g \in \mathcal{X}_+$ and any $t \geq 0$.*

Proof. We introduce a dual problem of (2.26) defined on the space $C_0(\mathbb{R})$ by

$$\partial_t \varphi = \tilde{\Lambda} \varphi = \tilde{\mathcal{B}} \varphi + \tilde{\mathcal{A}} \varphi, \quad \varphi(0, \cdot) = \varphi_0, \quad (2.28)$$

where the operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are defined by

$$\tilde{\mathcal{B}} \varphi = \partial_x \varphi - a(x, 0) \varphi, \quad \tilde{\mathcal{A}} \varphi = a(x, 0) \varphi(0).$$

A solution φ to (2.28) then satisfies

$$\varphi(t) = S_{\tilde{\mathcal{B}}}(t) \varphi_0 + (S_{\tilde{\mathcal{B}}} * \tilde{\mathcal{A}} \varphi)(t).$$

Let us fix $\varphi_0 \in C_0(\mathbb{R})$ such that $\varphi_0 \leq 0$ and let us prove that $\varphi(t) \leq 0$ for any $t \geq 0$. We obviously have that $S_{\tilde{\mathcal{B}}}$ is a positive operator and it is a contraction in $C_0(\mathbb{R})$. Taking the positive part in (2.28), we get

$$\begin{aligned} \varphi_+(t) &\leq S_{\tilde{\mathcal{B}}}(t) \varphi_{0+} + (S_{\tilde{\mathcal{B}}} * \tilde{\mathcal{A}} \varphi_+)(t) \\ &\leq a_1 \int_0^t S_{\tilde{\mathcal{B}}}(t-s) \varphi_+(0) \, ds, \end{aligned}$$

so that

$$\|\varphi_+(t)\|_{L^\infty} \leq C \int_0^t \|\varphi_+(s)\|_{L^\infty} \, ds.$$

From Grönwall's lemma, we deduce that $\varphi_+(t) = 0$ for any $t \geq 0$ and then $\varphi \leq 0$. We conclude by observing that S_{Λ_0} is the dual of $S_{\tilde{\Lambda}}$. \square

Lemma 2.6. *$-\Lambda_0$ satisfies the following version of the strong maximum principle: for any given $g \in \mathcal{X}_+$ and $\mu \in \mathbb{R}$, there holds*

$$g \in D(\Lambda_0) \setminus \{0\} \text{ and } (-\Lambda_0 + \mu)g \geq 0 \text{ imply } g > 0.$$

Proof. Suppose that there holds $(-\Lambda_0 + \mu)g \geq 0$ for g satisfying the above conditions. It is only necessary to prove that g does not vanish in \mathbb{R}_+ . Since $g \not\equiv 0$, there exists $x^* \in \mathbb{R}_+$ such that $g(x^*) > 0$. Rewriting the assumption as

$$\partial_x g + (a(x, 0) + \mu)g \geq \delta_{x=0} \int_0^\infty a(x, 0)g dx,$$

we observe that

$$\partial_x (e^{A(x,0)+\mu x} g) = e^{A(x,0)+\mu x} (\partial_x g + (a(x, 0) + \mu)g) \geq 0. \quad (2.29)$$

(i) For $x \in (x^*, \infty)$, we have

$$e^{A(x,0)+\mu x} g \geq e^{A(x^*,0)+\mu x^*} g(x^*) > 0.$$

(ii) For $x \in (0, x^*)$, by integrating the same equation on $(0, x)$, we obtain

$$\begin{aligned} e^{A(x,0)+\mu x} g &\geq \int_0^x \delta_{u=0} e^{A(u,0)+\mu u} \int_0^\infty a(y, 0)g(y) dy du + g(0) \\ &\geq \int_0^\infty a(y, 0)g(y) dy. \end{aligned}$$

From the positivity assumption (1.7) on a and step (i), we have

$$\int_0^\infty a(y, 0)g(y) dy > \frac{a_0}{2} \int_{\max\{x_0, x^*\}}^\infty g(y) dy > 0.$$

Therefore, g does not vanish on $(0, \infty)$. \square

Proof of Theorem 2.4. First, we know from Theorem 1.1 that there exists at least one nonnegative and non-vanishing solution F_0 to the eigenvalue problem $\Lambda_0 F_0 = 0$ and the associated dual eigenvector is $\psi = 1$. Next, we observe that, defining the $\text{sign} f$ operator for $f \in D(\Lambda_0^2)$ by

$$[(\text{sign} f)^* \psi](x) := \frac{1}{2|f(x)|} [\bar{f}(x)\psi(x) + f(x)\bar{\psi}(x)], \quad \forall \psi \in C_0(\mathbb{R}),$$

we have, for any $\psi \in C_0(\mathbb{R})_+$,

$$\begin{aligned} \Re \langle (\text{sign} f) \mathcal{M}_0[f], \psi \rangle &= \Re \langle \mathcal{M}_0[f], (\text{sign} f)^* \psi \rangle \\ &= \Re \left[\int a_0 f dx \right] \frac{\Re f(0)}{|f(0)|} \psi(0) \\ &\leq \int a_0 |f| dx \psi(0) = \langle \mathcal{M}_0[|f|], \psi \rangle, \end{aligned}$$

which is nothing but the complex Kato's inequality

$$\forall f \in D(\Lambda_0^2), \quad \Re(\text{sign} f) \Lambda_0 f \leq \Lambda_0 |f|. \quad (2.30)$$

We also observe that $D(\Lambda_0^2) \subset C_b(0, \infty)$, and, as a consequence, $g \in D(\Lambda_0^2)$ and $|g| > 0$ implies $g > 0$ or $g < 0$. We then may use exactly the same argument as in [14, 10, Proof of Theorem 5.3] (see also [11]):

- Kato's inequality (2.30) and the strong maximum principle imply that the eigenvalue $\lambda = 0$ is simple and the associated eigenspace is $\text{Vect}(F_0)$;
- together with the fact that S_{Λ_0} is a positive semigroup, one deduces that $\lambda = 0$ is the only eigenvalue with nonnegative real part.

We conclude to the spectral gap estimate (2.27) for some $\alpha_0 \in (a^*, 0)$ with the help of Theorem 2.2. \square

2.4 Weak connectivity regime - exponential stability of the linearized equation

We extend the exponential stability property which holds for a vanishing connectivity to the weak connectivity regime thanks to a perturbation argument.

Theorem 2.7. *There exist some constants $\varepsilon_0 > 0$, $\alpha < 0$ and $C > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ there hold $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$\|S_{\Lambda_\varepsilon}(t)g_0\|_{\mathcal{X}} \leq Ce^{\alpha t} \|g_0\|_{\mathcal{X}}, \quad \forall t \geq 0, \quad (2.31)$$

for any $g_0 \in \mathcal{X}$, $\langle g_0 \rangle = 0$.

The proof follows the stability theory for semigroups developed in Kato's book [9] and revisited in [12, 23, 11, 15].

Proof of Theorem 2.7. With the definitions (2.4), (2.10) and (2.11) of \mathcal{M}_ε , \mathcal{A}_ε and \mathcal{B}_ε , we have

$$(\mathcal{B}_\varepsilon - \mathcal{B}_0)g = (a(x, 0) - a(x, \varepsilon M_\varepsilon))g$$

and

$$(\mathcal{A}_\varepsilon - \mathcal{A}_0)g = (\mathcal{M}_\varepsilon[g] - \mathcal{M}_0[g]) \delta_0 - \varepsilon(\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \mathcal{M}_\varepsilon[g].$$

Together with the smoothness assumption (1.8), we deduce

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathcal{B}(\mathcal{X})} + \|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathcal{B}(\mathcal{X})} \leq C\varepsilon, \quad \forall \varepsilon \geq 0. \quad (2.32)$$

We then argue similarly as in the proof of [23, Theorem 2.15] (see also [9, 12, 11, 15]) and therefore just sketch the proof. We now define

$$K_\varepsilon(z) := (R_{\mathcal{B}_\varepsilon}(z)\mathcal{A}_\varepsilon)^2 R_{\Lambda_0}(z)(\Lambda_\varepsilon - \Lambda_0) \in \mathcal{B}(\mathcal{X}, X)$$

and we take any $\alpha \in (\alpha_0, 0)$, recalling that $a^* < \alpha_0 < 0$. We deduce from (2.32) and the estimates (i) and (ii) in Lemma 2.3 that for some $\eta < |\alpha|$, $\varepsilon_0 > 0$, small enough, $C > 0$, and for any $z \in \Delta_\alpha \setminus B(0, \eta)$, $\varepsilon \in [0, \varepsilon_0)$, we have

$$\|K_\varepsilon(z)\|_{\mathcal{B}(X)} \leq C\varepsilon < C\varepsilon_0 < 1, \quad (2.33)$$

and thus $(1 - K_\varepsilon(z))^{-1}$ is well defined in $\mathcal{B}(X)$. Using the elementary identities

$$R_{\Lambda_\varepsilon} = R_{\mathcal{B}_\varepsilon} - R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon} + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{\Lambda_\varepsilon} =: \mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{\Lambda_\varepsilon}, \quad (2.34)$$

and

$$R_{\Lambda_\varepsilon} = R_{\Lambda_0} + R_{\Lambda_0}(\Lambda_\varepsilon - \Lambda_0)R_{\Lambda_\varepsilon},$$

we get

$$(I - K_\varepsilon)R_{\Lambda_\varepsilon} = \mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{\Lambda_0}.$$

Because both terms are well defined in $\mathcal{B}(X)$ from Lemma 2.3 and the link between resolvent and semigroup, we also have

$$(I - K_\varepsilon)R_{L_\varepsilon} = \mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{L_0},$$

from what we deduce

$$R_{L_\varepsilon} = (I - K_\varepsilon)^{-1}(\mathcal{U}_\varepsilon + (R_{\mathcal{B}_\varepsilon}\mathcal{A}_\varepsilon)^2 R_{L_0}).$$

Since the RHS expression is clearly uniformly bounded in $\mathcal{B}(X)$ on the complex set $\Delta_\alpha \setminus B(0, \eta)$ for any $\varepsilon \in [0, \varepsilon_0)$, we have $\Sigma(L_\varepsilon) \cap \Delta_\alpha \subset B(0, \eta)$.

Using the definition of the eigenprojector Π_0 on the eigenspace associated to the spectral values of Λ_0 lying in $B(0, \eta)$ by mean of Dunford integral (see [9, Section III.6.4] or [8, 11]) and the analogous of (2.34) in $\mathcal{B}(X)$, we have

$$\begin{aligned} \Pi_0 &= \frac{i}{2\pi} \int_{|z|=\eta} R_{L_0}(z) dz \\ &= \frac{i}{2\pi} \int_{|z|=\eta} (R_{\mathcal{B}_0}\mathcal{A}_0)^2 R_{L_0} dz, \end{aligned}$$

by using that the contribution of holomorphic functions vanish. In a similar way, we have

$$\begin{aligned}\Pi_\varepsilon &= \frac{i}{2\pi} \int_{|z|=\eta} (I - K_\varepsilon + K_\varepsilon) R_{L_\varepsilon} dz \\ &= \frac{i}{2\pi} \int_{|z|=\eta} (R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 R_{L_0} dz + \frac{i}{2\pi} \int_{|z|=\eta} K_\varepsilon R_{L_\varepsilon} dz.\end{aligned}$$

For $g \in X$, we next compute

$$\begin{aligned}\|(\Pi_\varepsilon - \Pi_0)g\|_X &\leq \frac{1}{2\pi} \int_{|z|=\eta} \|((R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 - (R_{\mathcal{B}_0} \mathcal{A}_0)^2) R_{\Lambda_0} g\|_X dz \\ &\quad + \frac{1}{2\pi} \int_{|z|=\eta} \|K_\varepsilon R_{\Lambda_\varepsilon} g\|_X dz \leq C \varepsilon \|g\|_X,\end{aligned}$$

where we have used the identity

$$\begin{aligned}(R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon)^2 - (R_{\mathcal{B}_0} \mathcal{A}_0)^2 &= R_{\mathcal{B}_\varepsilon} \mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon} \{(\mathcal{A}_\varepsilon - \mathcal{A}_0) + (\mathcal{B}_0 - \mathcal{B}_\varepsilon) R_{\mathcal{B}_0} \mathcal{A}_0\} \\ &\quad + R_{\mathcal{B}_\varepsilon} \{(\mathcal{A}_\varepsilon - \mathcal{A}_0) + (\mathcal{B}_0 - \mathcal{B}_\varepsilon) R_{\mathcal{B}_0} \mathcal{A}_0\} R_{\mathcal{B}_0} \mathcal{A}_0\end{aligned}$$

and the estimates (2.33) and (2.32). As a consequence, we deduce

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathcal{B}(X)} < 1.$$

for any $\varepsilon \in (0, \varepsilon_0)$, up to take a smaller real number $\varepsilon_0 > 0$. From the classical result [9, Section I.4.6] (or more explicitly [23, Lemma 2.18]), we deduce that there exists $\xi_\varepsilon \in \Delta_\alpha$ such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\}, \quad \xi_\varepsilon \text{ is a simple eigenvalue,}$$

for any $\varepsilon \in [0, \varepsilon_0]$. We conclude by observing that $\xi_\varepsilon = 0$ because $1 \in X'$ and $\Lambda_\varepsilon^* 1 = 0$ (which is nothing but the mass conservation). \square

2.5 Weak connectivity regime - nonlinear exponential stability

Now, we focus on the nonlinear exponential stability of the solution to the evolution equation (1.1)–(1.3) in the case without delay. We start with an auxiliary result. We define the function $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi[g, \mu] := \int_0^\infty a(x, \varepsilon \mu) g(x) dx - \mu.$$

We denote by W_1 the optimal transportation Monge-Kantorovich-Wasserstein distance on the probability measures set $\mathbf{P}(\mathbb{R}_+)$ associated to the distance $d(x, y) = |x - y| \wedge 1$, or equivalently defined by

$$\forall f, g \in \mathbf{P}(\mathbb{R}_+), \quad W_1(f, g) := \sup_{\varphi, \|\varphi\|_{W^{1,\infty}} \leq 1} \int_0^\infty (f - g) \varphi.$$

Lemma 2.8. *Assume (1.8). There exists $\varepsilon_0 > 0$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\varphi_\varepsilon : \mathbf{P}(\mathbb{R}) \rightarrow \mathbb{R}$ which is Lipschitz continuous for the weak topology of probability measures and such that $\mu = \varphi_\varepsilon[g]$ is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g, \mu) = 0.$$

Proof of Lemma 2.8. Step 1. Existence. For any $g \in \mathbf{P}(\mathbb{R})$ we have $\Phi(g, 0) > 0$, and for any $g \in \mathbf{P}(\mathbb{R})$ and $\mu \geq 0$, we have

$$\Phi(g, \mu) \leq \|a\|_{L^\infty} - \mu,$$

so that $\Phi(g, \mu) < 0$ for $\mu > \|a\|_{L^\infty}$. By the intermediate value theorem and the continuity property of Φ , for any fixed $g \in \mathbf{P}(\mathbb{R}_+)$ and $\varepsilon \geq 0$, there exists at least one solution $\mu \in (0, \|a\|_{L^\infty}]$ to the equation $\Phi(g, \mu) = 0$.

Step 2. Uniqueness and Lipschitz continuity. Fix $f, g \in \mathbf{P}(\mathbb{R}_+)$ and consider $\mu, \nu \in \mathbb{R}_+$ such that

$$\Phi(f, \mu) = \Phi(g, \nu) = 0.$$

We have

$$\nu - \mu = \int_0^\infty a(x, \varepsilon \nu)(g - f) + \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu))f,$$

with

$$\left| \int_0^\infty a(x, \varepsilon \nu)(g - f) \right| \leq \|a(\cdot, \varepsilon \nu)\|_{W^{1,\infty}} W_1(g, f),$$

and

$$\left| \int_0^\infty (a(x, \varepsilon \nu) - a(x, \varepsilon \mu))f \right| \leq \|a(\cdot, \varepsilon \nu) - a(\cdot, \varepsilon \mu)\|_{L^\infty} \leq \varepsilon \|\partial_\mu a\|_{L^\infty} |\mu - \nu|.$$

We then obtain

$$|\mu - \nu| (1 - \varepsilon \|\partial_\mu a\|_{L^\infty}) \leq \|a(\cdot, \varepsilon \nu)\|_{W^{1,\infty}} W_1(g, f), \quad (2.35)$$

and we may fix $\varepsilon_0 > 0$ such that $1 - \varepsilon_0 \|\partial_\mu a\|_{L^\infty} \in (0, 1)$, $\varepsilon \in [0, \varepsilon_0]$. On the one hand, for $f = g$, we deduce that $\mu = \nu$ and that uniquely defines the mapping $\varphi_\varepsilon[g] := \mu$. On the other hand, the function is Lipschitz continuous because of (2.35). \square

We also recall the following classical Grönwall's type lemma.

Lemma 2.9. *Assume that $u \in C([0, \infty); \mathbb{R}_+)$ satisfies the integral inequality*

$$u(t) \leq C_1 e^{at} u_0 + C_2 \int_0^t e^{a(t-s)} u(s)^2 ds, \quad \forall t > 0,$$

for some constants $C_1 \geq 1$, $C_2, u_0 \geq 0$ and $a < 0$. Under the smallness assumption

$$a + 2C_2 u_0 < 0,$$

there holds

$$u(t) \leq \left(1 + \frac{C_1 u_0 C_2}{|a + 2C_2 u_0|}\right) C_1 e^{at} u_0, \quad \forall t \geq 0.$$

Proof of Lemma 2.9. We fix $A \in (C_1 u_0, 2C_1 u_0)$, so that $C_1 u(t) \leq A$ at least on a small interval, that is for any $t \in [0, \tau]$, $\tau > 0$ small enough, and then the integral inequality implies on the same interval

$$u(t) \leq C_1 e^{at} u_0 + C_2 C_1^{-1} A \int_0^t e^{a(t-s)} u(s) ds.$$

The classical Grönwall's lemma (for linear integral inequality) and the smallness assumption $a + C_2 C_1^{-1} A \leq 0$ imply

$$u(t) \leq C_1 u_0 e^{(a + C_2 C_1^{-1} A)t} \leq C_1 u_0 < A$$

on that interval. By a continuity argument, the first above inequality holds on \mathbb{R}_+ and then with $A := C_1 u_0$. Next, replacing that first estimate in the integral inequality we started with, we get

$$u(t) \leq C_1 e^{at} u_0 + C_2 C_1^2 u_0^2 e^{at} \int_0^t e^{(a + 2C_2 u_0)s} ds, \quad \forall t > 0,$$

from which we immediately conclude. \square

We come to the proof of our main result Theorem 1.2 in the case without delay.

Proof of Theorem 1.2 in the case without delay. We split the proof into two steps.

Step 1. New formulation. We start giving a new formulation of the solutions to the evolution and stationary equations in the weak connectivity regime $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is defined in Lemma 2.8. For a given initial datum $0 \leq f_0 \in L^1(\mathbb{R}_+)$ with unit mass the solution $f \in C([0, \infty); L^1(\mathbb{R}_+))$ to the evolution equation (1.1) and the solution F_ε to the stationary equation (1.5) clearly satisfy

$$\begin{aligned}\partial_t f + \partial_x f + a(\varepsilon \varphi[f])f &= 0, & f(t, 0) &= \varphi[f(t, \cdot)], \\ \partial_x F + a(\varepsilon M)F &= 0, & F(0) &= M = \varphi[F],\end{aligned}$$

where here and below the ε and x dependency is often removed without risk of misleading.

We introduce the variation function $g := f - F$ which satisfies the PDE

$$\partial_t g = -\partial_x g - a(\varepsilon M)g - \varepsilon a'(\varepsilon M)F \mathcal{M}[g] - Q[g] \quad (2.36)$$

with

$$Q[g] := a(\varepsilon \varphi[f])f - a(\varepsilon \varphi[F])F - a(\varepsilon \varphi[F])g - \varepsilon a'(\varepsilon \varphi[F])F \mathcal{M}[g],$$

where $\mathcal{M} = \mathcal{M}_\varepsilon$ is defined in (2.4). The above PDE is complemented with the boundary condition

$$g(t, 0) = \varphi[f(t, \cdot)] - \varphi[F],$$

and we may write again

$$\begin{aligned}\varphi[f] - \varphi[F] &= \int_0^\infty a(\varepsilon \varphi[f])f - \int_0^\infty a(\varepsilon \varphi[F])F \\ &= \int_0^\infty (a(\varepsilon M)g + \varepsilon a'(\varepsilon M)F \mathcal{M}[g]) + \int_0^\infty Q[g] \, dx \\ &= \mathcal{M}[g] + \mathcal{Q}[g], \quad \mathcal{Q}[g] := \langle Q[g] \rangle.\end{aligned}$$

As a consequence, we have proved that the variation function g satisfies the equation

$$\partial_t g = \Lambda_\varepsilon g + Z[g], \quad Z[g] := -Q[g] + \delta_0 \mathcal{Q}[g]. \quad (2.37)$$

Step 2. The nonlinear term. On the one hand, we obviously have

$$\langle Z[g] \rangle = 0, \quad \forall g \in M^1(\mathbb{R}_+). \quad (2.38)$$

On the other hand, in order to get an estimate on the nonlinear term $Z[g]$, we introduce the notation

$$\psi(u) = a(x, \varepsilon m_u) f_u,$$

where, for some fixed $g \in \mathbf{P}(\mathbb{R}_+)$, $\langle g \rangle = 0$, we have set

$$f := F + g, \quad f_u := uf + (1 - u)F, \quad m_u := \varphi[f_u].$$

We first notice that $\psi(0) = a(\varepsilon \varphi[F])F$ and $\psi(1) = a(\varepsilon \varphi[f])f$. Second, we have

$$\psi'(u) = a'_\varepsilon(m_u) f_u m'_u + a_\varepsilon(m_u) g. \quad (2.39)$$

In order to compute m'_u , we differentiate with respect to u the identity

$$m_u = \int_0^\infty a_\varepsilon(m_u) f_u dx,$$

and we have

$$m'_u = \int_0^\infty a'_\varepsilon(m_u) f_u dx m'_u + \int_0^\infty a_\varepsilon(m_u) g dx,$$

which implies

$$m'_u = \left(1 - \int_0^\infty a'_\varepsilon(m_u) f_u dx\right)^{-1} \int_0^\infty a_\varepsilon(m_u) g dx. \quad (2.40)$$

We may thus observe that $m'_0 = \mathcal{M}[g]$, so that $\psi'(0) = a'_\varepsilon(M) F \mathcal{M}_\varepsilon[g] + a_\varepsilon(M) g$, and therefore

$$Q[g] = \psi(1) - \psi(0) - \psi'(0).$$

Third, from (2.39), we have

$$\psi''(u) = a''_\varepsilon(m_u) f_u (m'_u)^2 + 2a'_\varepsilon(m_u) g m'_u + a'_\varepsilon(m_u) f_u m''_u,$$

and from (2.40), we have

$$\begin{aligned} m''(u) &= 2 \left(1 - \int_0^\infty a'_\varepsilon f_u\right)^{-2} \int_0^\infty a_\varepsilon g \int_0^\infty a'_\varepsilon g \\ &\quad + 2 \left(1 - \int_0^\infty a'_\varepsilon f_u\right)^{-3} \int_0^\infty a''_\varepsilon f \left(\int_0^\infty a_\varepsilon g\right)^2. \end{aligned}$$

In the small connectivity regime $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \|a'\|_\infty < 1$, we get the bound

$$\begin{aligned}
\|\psi''(u)\|_X &\leq \|a''_\varepsilon\|_\infty |m'_u|^2 + 2\|a'_\varepsilon\|_\infty \|g\|_X |m'_u| + \|a'_\varepsilon\|_\infty |m''_u| \\
&\leq \varepsilon^2 \frac{\|a''\|_\infty \|a\|_\infty^2}{(1 - \varepsilon \|a'\|_\infty)^2} \|g\|_X^2 + 2\varepsilon \frac{\|a'\|_\infty \|a\|_\infty}{1 - \varepsilon \|a'\|_\infty} \|g\|_X^2 \\
&\quad + 2\varepsilon^2 \frac{\|a'\|_\infty^2 \|a\|_\infty}{(1 - \varepsilon \|a'\|_\infty)^2} \|g\|_X^2 + 2\varepsilon^3 \frac{\|a''\|_\infty \|a'\|_\infty \|a\|_\infty}{(1 - \varepsilon \|a'\|_\infty)^3} \|g\|_X^2 \\
&\leq \varepsilon K \|g\|_X^2,
\end{aligned}$$

for some constant $K \in (0, \infty)$. Using the Taylor expansion

$$Q[g] = \psi(1) - \psi(0) - \psi'(0) = \int_0^1 (1-u) \psi''(u) du,$$

we then obtain

$$\|Z[g]\|_X \leq 2\|Q[g]\|_X \leq \int_0^1 (1-u) \|\psi''(u)\|_X du \leq C \|g\|_X^2.$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (2.37) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(f_0 - F) + \int_0^t S_{\Lambda_\varepsilon}(t-s) Z[g(s)] ds.$$

Using Theorem 2.7 and the second step, we deduce

$$\begin{aligned}
\|g(t)\|_X &\leq C e^{\alpha t} \|g_0\|_X + \int_0^t C e^{\alpha(t-s)} \|Z[g(s)]\|_X ds \\
&\leq C e^{\alpha t} \|g_0\|_X + C \varepsilon K \int_0^t e^{\alpha(t-s)} \|g(s)\|_X^2 ds,
\end{aligned}$$

for any $t \geq 0$ and for some constant $C \geq 1$, $\alpha < 0$, independent of $\varepsilon \in (0, \varepsilon_0]$. Observing that $\|g(t)\|_X \in C([0, \infty))$, we conclude thanks to Lemma 2.9. \square

3 Case with delay

This section is devoted to the proof of our main result, Theorem 1.2, in the case with delay by following the same strategy as in the case without delay

but adapting the functional framework. The main difference comes from the boundary term and it will be explained in the first subsection. We have already proved in Theorem 1.1 the existence of a unique stationary solution $(F_\varepsilon, M_\varepsilon)$ in the weak connectivity regime and we may then focus on the evolution equation.

3.1 Linearized equation and structure of the spectrum

In order to write as a time autonomous equation the linearized equation (1.12)-(1.13)-(1.14), we introduce the following intermediate evolution equation on a function $v = v(t, y)$

$$\partial_t v + \partial_y v = 0, \quad v(t, 0) = q(t), \quad v(0, y) = 0, \quad (3.1)$$

where $y \geq 0$ represents the local time for the network activity. That last equation can be solved with the characteristics method

$$v(t, y) = q(t - y) \mathbf{1}_{0 \leq y \leq t}.$$

Therefore, equation (1.14) on the variation $n(t)$ of network activity writes

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(dy),$$

and then equation (1.13) on the variation $q(t)$ of discharging neurons writes

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)],$$

with

$$\begin{aligned} \mathcal{O}_\varepsilon[g, v] &:= \mathcal{N}_\varepsilon[g] + \kappa_\varepsilon \mathcal{D}[v], \\ \mathcal{N}_\varepsilon[g] &:= \int_0^\infty a_\varepsilon(M_\varepsilon) g \, dx, \quad \kappa_\varepsilon := \int_0^\infty a'_\varepsilon(M_\varepsilon) F_\varepsilon \, dx. \end{aligned}$$

All together, we may rewrite the linear system (1.12)-(1.13)-(1.14), as the autonomous system

$$\partial_t (g, v) = \mathcal{L}_\varepsilon (g, v), \quad (3.2)$$

where the operator \mathcal{L}_ε is defined by

$$\mathcal{L}_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} := \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] \\ -\partial_y v \end{pmatrix},$$

with domain

$$D(\mathcal{L}_\varepsilon) := \{(g, v) \in W^{1,1}(\mathbb{R}_+) \times W^{1,1}(\mathbb{R}_+, \omega); g(0) = v(0) = \mathcal{O}_\varepsilon[g, v]\},$$

where $\omega(x) := e^{-\delta x}$ with $\delta > 0$ defining in (1.9). The associated semigroup $S_{\mathcal{L}_\varepsilon}(t)$ acts on

$$X := X_1 \times X_2 := L^1(\mathbb{R}_+) \times L^1(\mathbb{R}_+, \omega).$$

Considering the boundary condition as a source term, we also introduce the semigroup $S_{\Lambda_\varepsilon}(t)$ acting on

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+, \omega)$$

with the generator $\Lambda_\varepsilon = (\Lambda_\varepsilon^1, \Lambda_\varepsilon^2)$ given by

$$\begin{aligned} \Lambda_\varepsilon^1(g, v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v], \\ \Lambda_\varepsilon^2(g, v) &:= -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v]. \end{aligned}$$

In a similar way as in section 2.2, we have $S_{\Lambda_\varepsilon}|_X = S_{\mathcal{L}_\varepsilon}$.

As a first step, we establish that the semigroup S_{Λ_ε} has a nice decomposition structure with finite dimensional principal modes and a fast decaying remainder term.

Theorem 3.1. *Assume (1.6)-(1.7)-(1.8) and (1.9). The conclusions of Theorem 2.2 holds true with $a^\sharp := \max\{a^*, -\delta\} < 0$.*

The result is obtained as a consequence of the Spectral Mapping and Weyl's theory developed in [14, 10, 11] and taken over in section 2.2. For that purpose, we introduce the convenient splitting of the operator Λ_ε on \mathcal{X} as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ defined by

$$\mathcal{B}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{B}_\varepsilon^1(g, v) \\ \mathcal{B}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g, v) = \begin{pmatrix} \mathcal{A}_\varepsilon^1(g, v) \\ \mathcal{A}_\varepsilon^2(g, v) \end{pmatrix} = \begin{pmatrix} -a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix}.$$

It is only necessary to establish the following adequate properties of the operators \mathcal{A}_ε and \mathcal{B}_ε . We skip the rest of the proof and refer to the proof of Theorem 2.2 for more details.

Lemma 3.2. *Assume that a satisfies conditions (1.6)–(1.8) and that b satisfies (1.9). For any $\varepsilon \geq 0$, the operators \mathcal{A}_ε and \mathcal{B}_ε satisfy :*

- (i) $\mathcal{A}_\varepsilon \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+) \times W^{-1,1}(\mathbb{R}_+, \omega), \mathcal{X})$.
- (ii) $S_{\mathcal{B}_\varepsilon}(t)$ is a^\sharp -hypodissipative in both X and \mathcal{X} ;
- (iii) the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathcal{B}(\mathcal{X}, Y)} \leq C e^{at}, \quad \forall a > a^\sharp,$$

for some constant $C_a > 0$ and with $Y := Y_1 \times Y_2$, where $Y_1 = BV(\mathbb{R}_+) \cap L^1_1(\mathbb{R}_+)$ and $Y_2 = BV(\mathbb{R}_+, \omega) \cap L^1_1(\mathbb{R}_+, \omega)$.

Proof. (i) It is an immediate consequence of the fact that $\mathcal{D} \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+, \omega), \mathbb{R})$ and $\mathcal{N}_\varepsilon \in \mathcal{B}(W^{-1,1}(\mathbb{R}_+), \mathbb{R})$ because of (1.8) and (1.9).

(ii) Since $S_{\mathcal{B}_\varepsilon^1}$ is nothing but the semigroup $S_{\mathcal{B}_\varepsilon}$ defined in (2.12) which is a^* -dissipative thanks to Lemma 2.3-(ii), we just have to prove the dissipativity of the translation semigroup $S_{\mathcal{B}_\varepsilon^2}$ which is defined through the explicit formula $[S_{\mathcal{B}_\varepsilon^2}(t)v](y) = v(y-t)\mathbf{1}_{y-t \geq 0}$. That follows from

$$\|S_{\mathcal{B}_\varepsilon^2}(t)v\|_{X_2} \leq \int_0^\infty |v(y-t)|\mathbf{1}_{y-t \geq 0} e^{-\delta y} dy \leq e^{-\delta t} \|v\|_{X_2},$$

for any $v \in X_2$ and any $t \geq 0$.

(iii) For $(g, v) \in \mathcal{X}$, we have

$$\mathcal{A}^1 S_{\mathcal{B}}(t)(g, v)(x) = \gamma(x)D(t) + \delta_{x=0}N(t), \quad (3.3)$$

$$\mathcal{A}^2 S_{\mathcal{B}}(t)(g, v)(y) = \delta_{y=0}\kappa D(t) + \delta_{y=0}N(t), \quad (3.4)$$

with

$$\begin{aligned} \gamma(x) &:= \kappa \delta_{x=0} - a'(x)F_\varepsilon(x), \\ N(t) &:= \mathcal{N}[S_{\mathcal{B}^1}(t)g] = \int_0^\infty a(x)e^{A(x-t)-A(x)}g(x-t)\mathbf{1}_{x-t \geq 0}dx, \\ D(t) &:= \mathcal{D}[S_{\mathcal{B}^2}(t)v] = \int_0^\infty v(y-t)\mathbf{1}_{y-t \geq 0}b(dy), \end{aligned}$$

where here and below the ε dependency is removed without risk of ambiguity. We observe that

$$\begin{aligned} |N(t)| &\leq C a_1 e^{3\beta t} \|g\|_{\mathcal{X}_1}, \\ |D(t)| &\leq C e^{\delta t} \|v\|_{X_2}, \end{aligned}$$

for any $t \geq 0$. We then compute

$$\begin{aligned} N'(t) &= \int_0^\infty \partial_x [a(x)e^{-A(x)}] e^{A(x-t)} g(x-t) \mathbf{1}_{x-t \geq 0} dx \\ &= \int_0^\infty [a'(x) - a(x)^2] e^{A(x-t)-A(x)} g(x-t) \mathbf{1}_{x-t \geq 0} dx, \\ D'(t) &= \int_0^\infty v(y-t) \mathbf{1}_{y-t \geq 0} b'(dy) = b' * \check{v}(t), \end{aligned}$$

from what we get the estimates

$$\begin{aligned} |N'(t)| &\lesssim e^{a^* t} \|g\|_{\mathcal{X}_1}, \\ |D'(t)| &\lesssim e^{-\delta t} \|v\|_{\mathcal{X}_2}. \end{aligned}$$

Denoting

$$\begin{aligned} T_1(t)(g, v)(x) &:= (S_{\mathcal{B}^1} * \mathcal{A}^1 S_{\mathcal{B}})(t)(g, v)(x), \\ T_2(t)(g, v)(y) &:= (S_{\mathcal{B}^2} * \mathcal{A}^2 S_{\mathcal{B}})(t)(g, v)(y), \end{aligned}$$

we compute

$$\begin{aligned} T_1(t)(g, v)(x) &= \int_0^t S_{\mathcal{B}^1}(s) (\gamma(x) D(t-s) + \delta_{x=0} N(t-s)) ds \\ &= \int_0^t e^{A(x-s)-A(x)} (\gamma(x-s) D(t-s) + \delta_{x-s=0} N(t-s) \mathbf{1}_{x-s \geq 0}) ds \\ &= e^{-A(x)} (\nu * \check{D}_t)(x) + e^{-A(x)} N(t-x) \mathbf{1}_{x \leq t}, \\ T_2(t)(g, v)(y) &= \int_0^t S_{\mathcal{B}^2}(s) \delta_{y=0} (\kappa D(t-s) + N(t-s)) ds \\ &= \int_0^t \delta_{y-s=0} (\kappa D(t-s) + N(t-s)) ds \\ &= (\kappa D(t-y) + N(t-y)) \mathbf{1}_{y \leq t}, \end{aligned}$$

where we use the notation $\nu := \gamma e^A$. We next differentiate the above identity, and we get

$$\begin{aligned} \partial_x T_1(t)(g, v)(x) &= -a(x) e^{-A(x)} ((\nu * \check{D}_t)(x) + N(t-x) \mathbf{1}_{x \leq t}) \\ &\quad - e^{-A(x)} (\nu * \check{D}'_t)(x) + N'(t-x) \mathbf{1}_{x \leq t} \\ &\quad - e^{-A(x)} \nu(x-t) D(0) \mathbf{1}_{x-t \geq 0} - e^{-A(x)} N(0) \delta_{x=t} \\ &\quad + e^{-A(x)} \nu(x) D(t), \\ \partial_y T_2(t)(g, v)(y) &= -(\kappa D' + N')(t-y) \mathbf{1}_{y \leq t} - (\kappa D + N)(0) \delta_{y=t} \end{aligned}$$

All together, we deduce

$$\begin{aligned}\|\partial_x T_1(t)(g, v)(x)\|_{X_1} &\lesssim e^{a^\sharp t} \|(g, v)\|_{\mathcal{X}}, \\ \|\partial_y T_2(t)(g, v)(y)\|_{X_2} &\lesssim e^{a^\sharp t} \|(g, v)\|_{\mathcal{X}},\end{aligned}$$

and the similar estimate for $\|(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)(g, v)\|_{\mathcal{X}}$. As a consequence, the announced estimate holds for the family of operators $S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}}$. \square

3.2 The vanishing connectivity regime

When the network connectivity parameter vanishes, $\varepsilon = 0$, the linearized operator simplifies as

$$\Lambda_0 \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x, 0)g + \delta_{x=0} \mathcal{O}_0[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_0[g, v] \end{pmatrix}, \quad (3.5)$$

where $\mathcal{O}_0[g, v] = \mathcal{N}_0[g] = \int_0^\infty a(x, 0)g(x)dx$. The associated semigroup is exponentially stable as shown in the following theorem.

Theorem 3.3. *There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\Lambda_0) \cap \Delta_\alpha = \{0\}$ and for any $(g_0, v_0) \in \mathcal{X}$, $\langle g_0 \rangle = 0$, there holds*

$$\|S_{\Lambda_0}(t)(g_0, v_0)\|_{\mathcal{X}} \leq C e^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}}, \quad \forall t \geq 0. \quad (3.6)$$

Proof of Theorem 3.3. Since $\Lambda_0^1 = \Lambda_0$, from Theorem 2.4 we have already proved that $g(t) := S_{\Lambda_0^1}(t)g_0$ satisfies $\|g(t)\|_{X_1} \leq C e^{at} \|g_0\|_{X_1}$ for any $t \geq 0$ and any $a \in (a^*, 0)$. We then focus on Λ_0^2 . The Duhamel formula associated to the equation $\partial_t v = \Lambda_0^2(g, v)$ writes

$$v(t) = S_{\mathcal{B}_0^2}(t)v_0 + \int_0^t S_{\mathcal{B}_0^2}(t-s) \mathcal{A}_0^2(g(s), v(s)) ds.$$

Using the already known estimate on $g(t)$, we deduce

$$\begin{aligned}\|S_{\Lambda_0^2} v_0(t)\|_{X_2} &= \|v(t)\|_{X_2} \leq \|S_{\mathcal{B}_0^2}(t)v_0\|_{X_2} + \int_0^t \|S_{\mathcal{B}_0^2}(t-s) \delta_0 \mathcal{N}_0[g(s)]\|_{X_2} ds \\ &\leq e^{-\delta t} \|v_0\|_{X_2} + \int_0^t e^{-\delta(t-s)} C e^{as} \|g_0\|_{X_1} ds \\ &\leq C e^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}}\end{aligned}$$

for $\max\{a, -\delta\} < \alpha < 0$, which yields our conclusion. \square

3.3 Weak connectivity regime - exponential stability of the linearized equation

In this part, we shall discuss the geometry structure of the spectrum of the linearized time elapsed equation in weak connectivity regime taking delay into account and using again a perturbation argument.

Theorem 3.4. *There exists some constants $\varepsilon_0 > 0$, $C \geq 1$ and $\alpha < 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ there holds $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$\|S_{\Lambda_\varepsilon}(t)(g_0, v_0)\|_{\mathcal{X}} \leq Ce^{\alpha t} \|(g_0, v_0)\|_{\mathcal{X}}, \quad (3.7)$$

for any $(g_0, v_0) \in \mathcal{X}$ such that $\langle g_0 \rangle = 0$.

We start presenting a technical result needed in the proof below.

Lemma 3.5. *The operator Λ_ε is continuous with respect to ε , and more precisely*

$$\|\Lambda_\varepsilon - \Lambda_0\|_{\mathcal{B}(\mathcal{X})} \leq O(\varepsilon). \quad (3.8)$$

Proof. For all $(g, v) \in \mathcal{X}$, we have

$$\Lambda_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g, v] \end{pmatrix}, \quad (3.9a)$$

$$\Lambda_0 \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x, 0)g + \delta_{x=0} \mathcal{O}_0[g, v] \\ -\partial_y v + \delta_{y=0} \mathcal{O}_0[g, v] \end{pmatrix}. \quad (3.9b)$$

From (3.9a)-(3.9b), we deduce

$$(\Lambda_\varepsilon - \Lambda_0) \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} (a(x, 0) - a_\varepsilon)g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0}(\mathcal{O}_\varepsilon[g, v] - \mathcal{O}_0[g, v]) \\ \delta_{y=0}(\mathcal{O}_\varepsilon[g, v] - \mathcal{O}_0[g, v]) \end{pmatrix}.$$

We then compute

$$\begin{aligned} \|(\Lambda_\varepsilon - \Lambda_0)(g, v)\|_{\mathcal{X}} &= \|(a(x, 0) - a_\varepsilon)g\|_{\mathcal{X}_1} + \|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{\mathcal{X}_1} + 2\|\mathcal{O}_\varepsilon[g, v] - \mathcal{O}_0[g, v]\| \\ &\leq 3\|(a_\varepsilon - a_0)g\|_{\mathcal{X}_1} + 2\|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{\mathcal{X}_2} \\ &\leq 3\varepsilon\|a'\|_\infty\|g\|_{\mathcal{X}_1} + 2\varepsilon a_1\|a'\|_\infty(1 - \varepsilon\|a'\|_\infty)\|F_\varepsilon\|_{\mathcal{X}_1}\|v\|_{\mathcal{X}_2} \\ &= C\varepsilon\|(g, v)\|_{\mathcal{X}}, \end{aligned}$$

which is nothing but (3.8). \square

Proof of Theorem 3.4. With the help of Lemma 3.5, we may proceed similarly as in the proof of Theorem 2.7 (see also again [23, 15]) and we conclude that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\},$$

with $|\xi_\varepsilon| \leq O(\varepsilon)$ and ξ_ε is algebraically simple. We observe that

$$\Lambda_\varepsilon^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_x \varphi - a_\varepsilon \varphi + a_\varepsilon(\varphi(0) + \psi(0)) \\ \partial_y \psi + \kappa_\varepsilon b \psi(0) + \kappa_\varepsilon b \varphi(0) - b \int a'_\varepsilon F_\varepsilon \varphi \, dx \end{pmatrix},$$

from which we deduce that $\Lambda_\varepsilon^*(1, 0) = 0$. Then $0 \in \Sigma(\Lambda_\varepsilon^*)$ and $\xi_\varepsilon = 0$. Moreover, the orthogonality condition $\langle g_0 \rangle = \langle (g_0, v_0), (1, 0) \rangle_{\mathcal{X}, \mathcal{X}'} = 0$ implies that the exponential estimate (3.7) holds. \square

3.4 Weak connectivity regime - nonlinear exponential stability

We finally come back on the nonlinear problem and we present the proof of the second part of our main result for the case with delay.

Proof of Theorem 1.2 in case with delay. We write the system as

$$\begin{aligned} \partial_t f &= -\partial_x f - a_\varepsilon(\mathcal{D}[u])f + \delta_0 \mathcal{P}[f, \mathcal{D}[u]] \\ \partial_t u &= -\partial_y u + \delta_0 \mathcal{P}[f, \mathcal{D}[u]], \end{aligned}$$

with

$$\mathcal{P}[f, m] = \int a(m)f, \quad \mathcal{D}[u] = \int bu.$$

We recall that the steady state (F, U) , $U := M\mathbf{1}_{y \geq 0}$, satisfies

$$\begin{aligned} 0 &= -\partial_x F - a_\varepsilon(M)F + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathcal{P}[F, \mathcal{D}[U]]. \end{aligned}$$

We introduce the variation $g := f - F$ and $v = u - U$. The equation on g is

$$\begin{aligned} \partial_t g &= -\partial_x g - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)F + \delta_0(\mathcal{P}[f, \mathcal{D}[u]] - \mathcal{P}[F, \mathcal{D}[U]]) \\ &= -\partial_x g - a_\varepsilon(M)f - a'_\varepsilon F \mathcal{D}[v] - Q[g, v] + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \Lambda_\varepsilon^1(g, v) + \mathcal{Z}^1[g, v], \end{aligned}$$

where

$$\begin{aligned}\mathcal{Q}[g, v] &:= \langle Q[g, v] \rangle, \\ \mathcal{Z}^1[g, v] &:= -Q[g, v] + \delta_0 \mathcal{Q}[g, v],\end{aligned}$$

with $Q[g, v]$ denoting that

$$\begin{aligned}Q[g, v] &:= a_\varepsilon(M)F - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)f + a'_\varepsilon F \mathcal{D}[v] \\ &= \Phi(0) - \Phi(1) + \Phi'(0),\end{aligned}$$

where

$$\Phi(\theta) = a_\varepsilon(\mathcal{D}[\theta u + (1 - \theta)U])(\theta f + (1 - \theta)F).$$

The equation on v is

$$\begin{aligned}\partial_t v &= -\partial_y v + \delta_0(\mathcal{P}[f, \mathcal{D}[u]] - \mathcal{P}[F, \mathcal{D}[U]]) \\ &= -\partial_y v + \delta_0 \mathcal{O}[g, v] + \delta_0 \mathcal{Q}[g, v] \\ &= \Lambda_\varepsilon^2(g, v) + \mathcal{Z}^2[g, v],\end{aligned}$$

where

$$\mathcal{Z}^2[g, v] := \delta_0 \mathcal{Q}[g, v].$$

We then write the associated Duhamel formula

$$(g(t), v(t)) = S_{\Lambda_\varepsilon}(t)(g_0, v_0) + \int_0^t S_{\Lambda_\varepsilon}(t - s) \mathcal{Z}[g(s), v(s)] \, ds.$$

Referring to Step 2 in the proof of Theorem 1.2 gives a similar estimate $\|\mathcal{Z}[g, v]\|_{\mathcal{X}} \leq C \|(g, v)\|_{\mathcal{X}}^2$, we then conclude the rest part of Theorem 1.2 in the case with delay. \square

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